# Metric Spaces and Topology
## Lecture 16

**Cor.** Let $X$ be a nonempty perfect Polish space, e.g. $\mathbb{R}$. There is no Baire meas. (as a subset of $X^2$) well-order $<$ of $X$. ($<$ is a subset of $X^2$.)

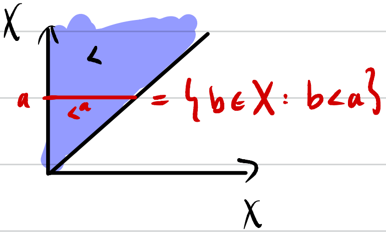
**Remark.** For top. spaces $X, Y$, there is a natural topology on $X \times Y$, called the product topology, where the open sets are generated by sets of the form $U \times V$ for $U \subseteq X$ and $V \subseteq Y$ open. In case $X, Y$ are metric spaces with metrics $d_X$ and $d_Y$, respectively, $X \times Y$ is a metric space with, for example, the $d_\infty$-metric, i.e.

$$d_{X\times Y}((x,y),(x',y')) := \max\{d_X(x,x'), d_Y(y,y')\}.$$

$\boxed{\text{HW}}$ In this case, the sets $U \times V$, $U \subseteq X$ and $V \subseteq Y$ open, form a basis for $X \times Y$. In particular, this holds for $X \times X =: X^2$.

**Proof of Corollary.** Suppose towards a contradiction that $\exists$ well-order

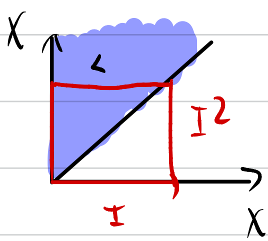
$<$ on $X$. Recall that $I \subseteq X$ is called *initial* if it's closed downward under $<$, i.e. if $a \in I$ then all $b < a$ are also in $I$ (in other words, $<_a \subseteq I$). $X$ itself is an initial set,




and other initial sets $I$ are of the form $<^a$ for some $a \in X$ (indeed, take $a$ to be the $<$-least element of $X \setminus I$). Note that $<^a \subsetneq <^b \iff a < b$, so the set of initial sets is also well-ordered under $\subsetneq$.

**Claim.** If an initial set $I \subseteq X$ is nonmeagre, then its restriction $<|_I := < \cap I^2$ is also nonmeagre.

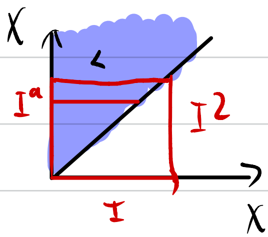
**Pf.** By Kuratowski–Ulam, $I^2$ is nonmeagre (each fiber of $I^2$ is equal to $I$, which is nonmeagre). But $I^2 = (<|_I) \cup (>|_I) \cup (=|_I)$, where $\Delta_I := \{(x,x) : x \in I\}$.




$\boxed{\text{HW}}$ $X$ is perfect $\iff$ $\Delta_X$ is nowhere dense.

The map $(x,y) \mapsto (y,x)$ is a homeomorphism of from $X^2$ to $X^2$, which maps $<|_I$ onto $>|_I$, so both are meagre/nonmeagre at the same time, so they must both be nonmeagre because $X$ is Baire. $\boxed{\text{Claim}}$

$X$ itself is an initial set and is nonmeagre (because $X$ is Baire) so we may take the least nonmeagre initial set $I$, using that $<$ is a well-order. Since each fiber $I^a$ of $I$ is still an initial set which is strictly contained in $I$, it must be meagre. By Kuratowski–Ulam, $<|_I$ is meagre, contradicting the claim. $\square$



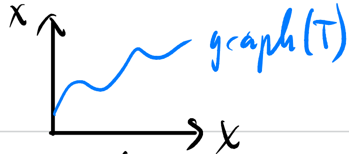

## Generic ergodicity and Borel graphs.

Let $X$ be a Polish space. We consider equivalence relations on $X$. They arise naturally from many sources, in particular, from graphs and from action of transformations or groups of transformations.

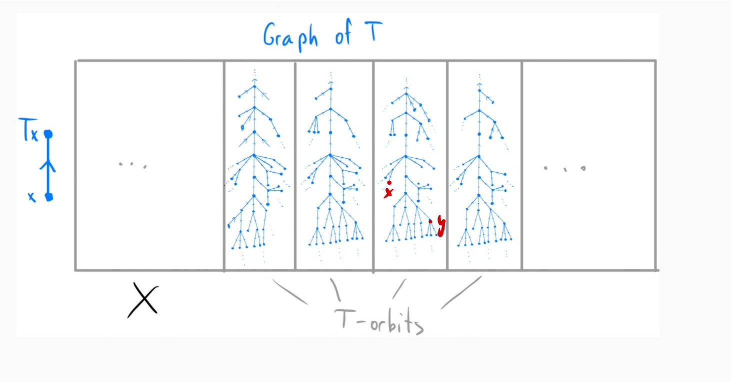
**Def.** We say that a transformation $T : X \to X$ is Borel (resp. Baire meas.) if the $T$-preimage of every open set is Borel (resp. Baire meas.). The graph of $T$ is

$$\operatorname{graph}(T) := \{(x, Tx) : x \in X\} \subseteq X^2.$$

We usually depict this as



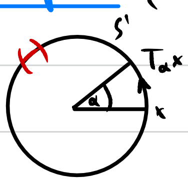

Modern combinatorial dynamics depicts graph(T) as:



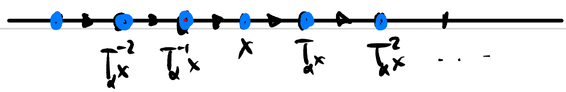

We denote this graph by $G_T$ and we call the $G_T$-connected components the orbits of $T$. The equivalence relation of being in the same connected component/orbit is called the orbit equivalence relation of $T$ and denoted by $E_T$.

Obs. $\forall x, y \in X$, $x E_T y \iff \exists n, m \in \mathbb{N}$ s.t. $T^n x = T^m y$.

Examples. (a) Let $X := S^1 :=$ unit circle in $\mathbb{R}^2$.




Fix an angle $\alpha$ and let $T_\alpha : X \to X$ be rotation by $\alpha$. This is a homeomorphism. We call $T_\alpha$ a rational rotation if $\alpha/2\pi$ is rational, otherwise we call it an irrational rotation. For a rational rotation, orbits are finite and have the same size (maybe $q$, where $\alpha/2\pi = p/q$ irreducible), and each connected component is a cycle. For an irrational rotation, each orbit is infinite and, in fact, dense. $\boxed{\text{HW}}$ Each connected component in this case is a $\mathbb{Z}$-line




The rotation $T_\alpha$ is the same (is isomorphic to) the map $x \mapsto x + \alpha/2\pi \pmod 1$, where "mod 1" is defined by taking the fractional part, i.e.

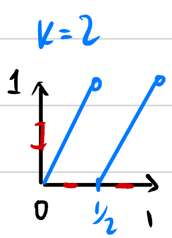
$[0,1) \to [0,1)$

$x + \alpha/2\pi \pmod 1$ := the fractional part of $x + \alpha/2\pi$.

In other words, we identify $[0,1)$ with $S^1$ via the map $x \mapsto e^{2\pi i x}$.


(b) The baker map $b_k : [0,1) \to [0,1)$

$$x \mapsto k \cdot x \pmod 1$$




This is a Borel map hence the preimages of open intervals are finite unions of (not necessarily open) intervals.

Each connected component is a complete binary tree rooted at $\infty$: $(k=2)$

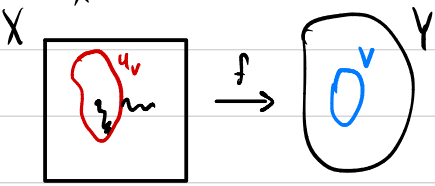

This transformation $b_k$ is isomorphic to the (left) shift transformation $\sigma_k : k^{\mathbb{N}} \to k^{\mathbb{N}}$. The isomorphism $[0,1) \to k^{\mathbb{N}}$
$$(x_n)_{n\in\mathbb{N}} \mapsto (x_{n+1})_{n\in\mathbb{N}}$$
is given by mapping $x \in [0,1)$ to its $k$-base representation.

For an eq. rel. $E$ on $X$, a set $Y \subseteq X$ is called $E$-invariant if it's a union of $E$-classes, equivalently, $x \in Y \Rightarrow [x]_E \subseteq Y$, where $[x]_E$ denotes the class of $X$.

**Prop.** Let $X, Y$ be metric (top.) spaces and suppose $Y$ is $2^{nd}$ ctbl. Then every Baire meas. function $f: X \to Y$ (e.g. Borel functions) is continuous when restricted to a comeagre set, i.e. $\exists$ comeagre $X' \subseteq X$ s.t. $f|_{X'} : X' \to Y$ is continuous.

**Proof.** Fix a ctbl basis $\mathcal{V}$ for $Y$ and note that a function to $Y$

is continuous if the preimages of basic open sets are open.
For every $V \in \mathcal{V}$, we know that $f^{-1}(V)$ is Baire meas. so
$\exists$ open set $U_V =^* f^{-1}(V)$. Letting $X' := X \setminus \bigcup_{V \in \mathcal{V}} \underbrace{(U_V \triangle f^{-1}(V))}_{\text{meagre}}$,
we see that $X'$ is comeagre and
$f|_{X'}^{-1}(V) = U_V \cap X'$ for each $V \in \mathcal{V}$, so $f|_{X'}$ is continuous. $\square$




**Def.** An equiv. rel. $E$ on $X$ is called *generically ergodic* if every $E$-invariant Baire measurable set is meagre or comeagre. We say a transformation $T : X \to X$ is *generically ergodic* if so is $E_T$.